CERTAIN ASPECTS FROM SIMILARITY THEORY APPLICABLE TO THE PROCESSES OF MOMENTUM TRANSFER

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Based on the "triple analogy" between the processes of heat transfer, diffusion, and momentum transfer, we introduce the concept of two independent similarity criteria to describe the hydrodynamics of fluid motion, analogous to the Nusselt and Stanton numbers.

The agreement between the equations describing the transfer processes for heat, mass, and momentum are referred to as the "triple analogy" [1]:

$$q = -\lambda \operatorname{grad} T = -a \frac{\partial (\rho c_p T)}{\partial y}; \tag{1}$$

$$j = -D \operatorname{grad} C; \tag{2}$$

$$\tau = -\mu \operatorname{grad} u = -\nu \frac{\partial (\rho u)}{\partial y}, \tag{3}$$

where a, D, and ν are the coefficients of molecular transfer. Particularly for turbulent flows, these generally do not lend themselves to analysis. We are therefore compelled to resort to empirical coefficients. The coefficients α and β are thus introduced into the processes of heat and mass transfer by means of the relationships

$$q = \alpha \Delta T, \tag{4}$$

$$j = \beta \Delta C. \tag{5}$$

It is not α, but the ratio $\alpha / c_p \rho$ that is the analog of β in heat transfer.

It follows from similarity theory that α and β can be used to find two different dimensionless numbers, i.e., the Nusselt number Nu and the Stanton number St.

For the heat-transfer process we will thus have

$$\mathrm{Nu} = \frac{\alpha L}{\lambda}, \quad \mathrm{St} = \frac{\alpha}{c_p \rho u}.$$

Correspondingly, for the mass-transfer process

$$\mathrm{Nu}_m = \frac{\beta L}{D}, \quad \mathrm{St}_m = \frac{\beta}{u}.$$

The introduction of such numbers is particularly convenient because the Nusselt number is constant for purely molecular transfer (particularly for laminar flows), while for turbulent flows it is the Stanton number that is virtually constant [1, 4], because of its rather weak dependence on Re.

Since a complete physical analogy exists between the transfer processes of heat, matter, and momentum, in analogy with α and β we introduce the coefficient of momentum transfer (the coefficient of external

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friction) γ according to the relationship

$$\tau = \gamma \Delta u. \tag{6}$$

In particular, for the conditions of the internal problem (flow in tubes) we have $\tau_0 = \gamma u_{av}$.

The coefficient γ is expressed in units of $[ML^{-2}T^{-1}]$, so that it is not the actual coefficient γ but the ratio γ/ρ that is the analog of β and $\alpha/c_p\rho$; this ratio is also expressed in units of linear velocity ($[LT^{-1}]$).

With the transfer coefficient introduced in this manner we form the corresponding criteria. The Nusselt number for the momentum transfer is $Nu_\tau = \gamma L/\mu$ and the Stanton number $St_\tau = \mu/\rho u$.

As follows from the conclusions of similarity theory, the Nusselt and Stanton numbers must be functions of the criteria characterizing the physical properties of the medium and the nature of the motion, i.e., the Prandtl (Pr) and the Reynolds (Re) numbers. Since the momentum-transfer process is independent of Pr ($Pr_\tau = \nu/\nu \equiv 1$), it follows that

$$Nu_\tau = f(Re), \quad St_\tau = f(Re).$$

We will determine this functional relationship on the example of the internal problem, thus associating the newly introduced criteria with the earlier employed hydraulic characteristics. Since it is possible to write that the tangential stress

$$\tau = -\mu \frac{du}{dr},$$

the frictional force F is equal to

$$F = -\mu S \frac{du}{dr},$$

where $S = 2\pi r_0 l$ is the friction surface, and the work of the frictional forces per unit length (per 1 kg of liquid flow)*

$$L_r = \frac{Fl}{fl\rho} = -\mu \frac{2\pi r_0 l l}{\pi r_0^2 l \rho} \frac{du}{dr} = \frac{2l}{r_0 \rho} \tau_0.$$

Hence

$$\tau_0 = \frac{L_r r_0 \rho}{2l}$$

or, using the hydraulic-resistance coefficient z,

$$\tau_0 = z\rho \frac{u_{av}^2}{8}.$$

Since by definition $\tau_0 = \gamma u_{av}$, we finally find that the momentum-transfer coefficient γ is equal to

$$\gamma = \frac{z}{8} \rho u_{av}.$$

Substituting this value into the expressions for Nu_τ and St_τ, we can write

$$Nu_\tau = \frac{z}{8} \frac{u_{av} d_0}{\nu} = \frac{z}{8} Re$$

and

$$St_\tau = \frac{z}{8}.$$

Since for a laminar flow we have

$$L_r = \frac{8\mu l}{\rho r_0^2} u_{av}$$

for a parabolic law of velocity distribution over the cross section [2], we find $\tau_0 = 4\mu u_{av}/r_0$ and $\gamma = 4\mu/r_0$.

*For simplicity of the consideration we will assume that the fluid is incompressible and exhibits a density that is constant over the cross section.

Consequently, for a laminar flow we have

$$\mathrm{Nu}_{\tau_{\mathrm{lam}}} = 8 = \mathrm{const},$$

which is what follows from the conclusions of similarity theory, and namely,

$$\mathrm{St}_{\tau_{\mathrm{lam}}} = 8/\mathrm{Re}.$$

Using this value, it is not difficult to find the familiar relationship for the hydraulic-resistance coefficient in the laminar regime:

$$z = \frac{64}{\mathrm{Re}}.$$

It is, in particular, from the constancy of St_{τ} in the regime of developed turbulence that we draw the conclusion that the coefficient of hydraulic resistance is constant and that it is independent of the flow regime.

Having introduced the conditional concept of a reduced film thickness, according to the relationship

$$\gamma = \mu/\delta,$$

we find

$$\delta = \frac{\mu}{\gamma} = \frac{d_0}{\mathrm{Nu}_{\tau}}.$$

For a laminar flow

$$\delta_{\mathrm{lam}} = \frac{r_0}{16}.$$

In the case of a developed turbulent flow the averaged velocity over the cross section varies logarithmically [3]:

$$\bar{u} = \sqrt{\frac{\tau_0}{\rho}}\,\frac{1}{\varkappa}\ln y + C.$$

Integration over the entire cross section, with consideration of the experimental data from [2, 3], yields

$$\bar{u}_{\mathrm{av}} = \sqrt{\frac{\tau_0}{\rho}}\left(1.94 + 5.75\lg\frac{11.5r_0}{\delta_{\mathrm{bl}}}\right).$$

Hence

$$\tau_0 = \gamma\bar{u}_{\mathrm{av}} = \frac{\rho\bar{u}_{\mathrm{av}}^2}{\left(1.94+5.75\lg\frac{11.5r_0}{\delta_{\mathrm{bl}}}\right)^2}.$$

The expressions for the dynamic analogs of Nu and St for turbulent flow will have the form

$$\mathrm{Nu}_{\tau_{\mathrm{turb}}} = \frac{\mathrm{Re}}{\left(1.94+5.75\lg\frac{11.5r_0}{\delta_{\mathrm{bl}}}\right)^2}$$

and

$$\mathrm{St}_{\tau_{\mathrm{turb}}} = \left(1.94+5.75\lg\frac{11.5r_0}{\delta_{\mathrm{bl}}}\right)^{-2} \approx \mathrm{const}.$$

Since

$$\delta_{\mathrm{bl}} = \frac{11.5d_0}{\mathrm{Re}\sqrt{\mathrm{St}_{\tau}}} = \frac{11.5\nu}{\bar{u}_{\mathrm{av}}\sqrt{\mathrm{St}_{\tau}}},$$

the velocity at the edge of the laminar layer can be defined as

$$u_\delta = \frac{\delta_{bl}}{d_0} \mathrm{Re}\, \mathrm{St}_\tau \bar{u}_{av} = 11.5\, \bar{u}_{av} \sqrt{\mathrm{St}_\tau}.$$

It is not difficult to show the procedure for the experimental determination of γ_{turb}, $\mathrm{Nu}_{\tau_{turb}}$, and $\mathrm{St}_{\tau_{turb}}$. Indeed, since [3]

$$\frac{\bar{u}_{max} - \bar{u}_{av}}{v_*} = D = \mathrm{const},$$

we find that

$$\tau_0 = \frac{\rho \bar{u}_{av}^2}{D^2} \left(\frac{\bar{u}_{max}}{\bar{u}_{av}} - 1 \right)^2.$$

Then, when D = 3.56, we have [2]

$$\gamma_{turb} = 0.0785\, \rho \bar{u}_{av} \left(\frac{\bar{u}_{max}}{\bar{u}_{av}} - 1 \right)^2,$$

$$\mathrm{Nu}_{\tau_{turb}} = 0.0785\ \mathrm{Re} \left(\frac{\bar{u}_{max}}{\bar{u}_{av}} - 1 \right)^2,$$

$$\mathrm{St}_{\tau_{turb}} = 0.0785 \left(\frac{\bar{u}_{max}}{\bar{u}_{av}} - 1 \right)^2.$$

The Nu_τ and St_τ introduced in this manner are also easily associated with the widely used resistance factor f defined by

$$\tau_0 = f \frac{\rho \bar{u}_{cp}^2}{2}.$$

Indeed, if by definition $\tau_0 = \gamma \bar{u}_{av}$, we have

$$\gamma = \frac{f}{2} \rho \bar{u}_{av}$$

and then

$$\mathrm{Nu}_\tau = \frac{f}{2} \mathrm{Re}, \quad \mathrm{St}_\tau = \frac{f}{2}.$$

Using the local turbulence theory [4], we can obtain the relationship between the criteria of heat and mass transfer and the criteria newly introduced in the form

$$\frac{1}{\mathrm{St}} = \frac{1}{\sqrt{\mathrm{St}_\tau}} \int_0^{\eta_0} \frac{d\eta}{\frac{1}{\mathrm{Pr}} + \frac{A(\eta)}{\nu}},$$

$$\frac{1}{\sqrt{\mathrm{St}_\tau}} = \int_0^{\eta_0} \frac{d\eta}{1 + \frac{A(\eta)}{\nu}}.$$

The variable η is introduced according to the relationship

$$\eta = \sqrt{\mathrm{St}_\tau} \frac{\bar{u}_{av}}{\nu} y.$$

Accordingly, from the Prandtl hypothesis, on a laminar sublayer of thickness δ_1, we will have

$$\mathrm{St}^{-1} = \mathrm{St}_\tau^{-1} + \frac{\bar{u}_{av}\delta_1}{\nu} (\mathrm{Pr} - 1) = \mathrm{St}_\tau^{-1} + \mathrm{Re}_{\delta_1} (\mathrm{Pr} - 1),$$

while according to the Landau and Levich theory [5, 6], for large values of the Prandtl number, we have

$$\mathrm{St}\, \mathrm{Pr}^{3/4} \approx \frac{2}{\pi} n \sqrt{2\, \mathrm{St}_\tau}.$$

In the last expression n is a universal empirical dimensionless constant.

Equating (3) and (6), we find

$$\gamma \Delta u = -\mu \frac{du}{dy},$$

from which, after simple transformations, we will have

$$\frac{\gamma L}{\mu} = \mathrm{Nu}_\tau = -\frac{du}{dy}\frac{L}{\Delta u} = -\frac{du}{\Delta u} \Big/ \frac{dy}{L} = -\frac{d\bar{u}}{d\bar{y}}$$

or

$$\mathrm{Nu}_\tau = -\operatorname{grad} \bar{u}.$$

Analogously we can write

$$\mathrm{Nu} = -\operatorname{grad} \bar{T}, \quad \mathrm{Nu}_m = -\operatorname{grad} \bar{C}.$$

Consequently, a general rule is the equality between the Nusselt number and the gradient of the corresponding field, in dimensionless coordinates.

We can thus see that the newly introduced dynamic analogies of the Nusselt and Stanton numbers for momentum transfer are well coordinated with system of earlier used criteria and make it possible uniquely to describe the transfer processes of heat, matter, and momentum in laminar and turbulent flows.

NOTATION

q	is the heat flux;
j	is the diffusion flux;
τ	is the tangential [shear] stress;
a	is the thermal diffusivity;
D	is the diffusion coefficient;
ν	is the kinematic viscosity;
λ	is the thermal conductivity;
μ	is the dynamic viscosity;
T	is the temperature;
C	is the concentration;
u	is the velocity;
ρ	is the density;
c_p	is the specific heat capacity;
α, β, γ	are the transfer coefficients for heat, matter, and momentum;
Nu	is the Nusselt number;
St	is the Stanton number;
Re	is the Reynolds number;
Pr	is the Prandtl number;
y, l, r, d	are linear dimensions;
F	is the force of surface friction;
L_r	is the work of the friction forces;
M	is the mass;
L	is a characteristic dimension;
l	is the length;
$\delta_{b.l}$	is the boundary-layer thickness;
av	denotes an average;
lam	denotes laminar flow;
turb	denotes turbulent flow;
max	denotes the maximum;
0	denotes the tube wall.

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